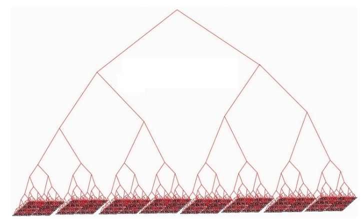
DESCRIPTIVE SET THEORY

Fall 2019 | Math 574-dst | Anush Tserunyan

TR 11am–12:20pm | 29 English Bldg

Classical, as well as contemporary, descriptive set theory (DST) is part of analysis, rather than set theory[1]. It combines techniques from analysis, topology, combinatorics, recursion theory, set theory, and other areas of mathematics to study *definable* subsets of (and functions on) Polish spaces (e.g. $\mathbb{R}^n$, $\mathbb{N}^{\mathbb{N}}$, $L^p(\mathbb{R})$). Examples of such sets include Borel, analytic (projections of Borel[2]), co-analytic (complement of analytic), etc. A typical example (perhaps the first) of a theorem in DST is Cantor's Perfect Set Theorem, which states that any uncountable Polish space (e.g. $\mathbb{R} \setminus \mathbb{Q}$) contains a Cantor set.



Cantor space

At its earlier stage, a central interest in DST was investigating the regularity properties of definable sets such as the perfect set property, measurability, and the Baire measurability. At the heart of this lies the theory of *infinite games*, which we will study.

For the past 30 years, a major focus of descriptive set theory has been the study of *equivalence relations* on Polish spaces that are definable when viewed as sets of pairs. Such equivalence relations arise naturally all over mathematics since many mathematical objects (such as Riemann surfaces, Banach spaces, measure-preserving transformations, etc.) can be parameterized as points in Polish spaces. Classifying these points (e.g. Banach spaces) up to some equivalence relation (e.g. isomorphism) means understanding the (Borel) complexity of this equivalence relation. DST provides a rigorous framework for this, as well as tools.

The theory of definable equivalence relations is intertwined with *actions of Polish groups* (e.g. all countable groups, Lie groups, many automorphism groups) and the combinatorics of definable (typically, analytic or Borel) *graphs on Polish spaces*, thus lying in the nexus of ergodic theory, topological dynamics, measured group theory, and combinatorics.

Prerequisites: Basic pointset topology and real analysis

Course material: A. Tserunyan, *Introduction to Descriptive Set Theory*, lecture notes [link]

[1] The name is a rather historical artifact.

[2] Lebesgue famously delayed the development of DST by a decade publishing a false proof that projections of Borel sets are Borel — they typically aren't!